Mathematics 222B Lecture 4 Notes

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1 Trace and Extension Theorems and Introduction to Sobolev Inequalities

Today, we will discuss

(i) trace and extension (from the boundary) theorems

(ii) Sobolev inequalities.

1.1 The trace theorem

Let U be an open subset of $\mathbb{R}^d$ with ∂U being C^1 and $1 < p < \infty$. Recall that for any integer $k \geq 0$, $C^\infty(\overline{U})$ is dense in $W^{k,p}$. In particular, $C^\infty(\overline{U})$ is dense in $W^{1,p}(U)$. Our aim is to discuss the restriction of $u \in W^{1,p}(U)$ to ∂U. Since the boundary is a measure 0 set, this is hard to specify directly (as L^p functions are only well-defined modulo null sets), so we will achieve this by appealing to the dense subset $C^\infty(\overline{U})$.

Definition 1.1. For $u \in C^1(\overline{U})$, we define the **trace** to be $\operatorname{tr}_{\partial U} u = u|_{\partial U}$.

We wish to extend this operation to all of $W^{1,p}(U)$. Note that $\operatorname{tr}_{\partial U}$ is linear, so we can extend it if we know it is bounded.

Theorem 1.1 (Trace theorem, non-sharp)**.** *Let U be a bounded, open subsets of $\mathbb{R}^d$ with C^1 boundary ∂U, and let $1 < p < \infty$. Then for $u \in C^1(\overline{U})$, we have*

$$\| \operatorname{tr}_{\partial U} u\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}.$$

(i) As a consequence, $\operatorname{tr}_{\partial U}$ is extended (uniquely) by continuity (and density of $C^1(\overline{U}) \subseteq W^{1,p}(U)$) to $\operatorname{tr}_{\partial U} : W^{1,p}(U) \to L^p(\partial U)$.

(ii) Moreover, $u \in W_0^{1,p}(U) \iff \operatorname{tr}_{\partial U} u = 0$.

Remark 1.1. The the map $\operatorname{tr}_{\partial U} : W^{1,p}(U) \to L^p(\partial U)$ is not surjective.

Instead of proving this theorem (and you can check the proof in section 5.5 of Evans' book), we will understand the sharp trace theorem in a restricted setting.

The setting we have in mind is $p = 2$. The advantage here is that we may use the theory of the Fourier transform and Plancherel's theorem. We will also focus on the domain $U = \mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^d > 0\}$ with boundary $\{(x', 0) \in \mathbb{R}^d\} \cong \mathbb{R}^{d-1}$, where $x' := (x^1, \dots, x^{d-1})$.

Recall the Fourier transform characterization of the H^k norm:

$$\|u\|^2_{H^k} \simeq \|(1 + |\xi|^2)^{k/2}\widehat{u}\|^2_{L^2_\xi}, \qquad k \geq 0 \text{ an integer.}$$

If we replace k with any $s \in \mathbb{R}$, we can talk about **fractional (L^2-based) Sobolev spaces**.

Theorem 1.2 (Sharp trace theorem)**.** *For $u \in C^1(\overline{\mathbb{R}^d_+}) \cap H^1(\mathbb{R}^d_+)$,*

$$\|\operatorname{tr}_{\partial U} u\|_{H^{1/2}(\mathbb{R}^{d-1})} \leq C\|u\|_{H^1(\mathbb{R}^d_+)}.$$

Proof. Take $u \in C^1(\overline{\mathbb{R}^d_+}) \cap H^1(\mathbb{R}^d_+)$. Using the extension procedure from last time, we can find a $\widetilde{u} \in C^1(\mathbb{R}^d)$ such that

$$\|\widetilde{u}\|_{H^1(\mathbb{R}^d)} \leq C\|u\|_{H^1(\mathbb{R}^d_+)}.$$

Then

$$\begin{aligned}
\operatorname{tr}_{\partial U} u(x') &= u(x', 0) \\
&= \widetilde{u}(x', 0) \\
&= \int \mathcal{F}_{x^d}\widetilde{u}(x', \xi_d)\frac{1}{2\pi}d\xi_d.
\end{aligned}$$

On the other hand,

$$\mathcal{F}_{x'} \operatorname{tr}_{\partial U} u(\xi') = \int \mathcal{F}\widetilde{u}(\xi', \xi_d)\frac{1}{2\pi}\, d\xi_d.$$

For now, let us not assume $s = 1/2$ so we can see where this choice comes from.

$$\begin{aligned}
\|\operatorname{tr}_{\partial U} u\|_{H^s(\mathbb{R}^{d-1})} &\sim \|(1 + |\xi'|^2)^{s/2}\mathcal{F}_{x'} \operatorname{tr} u(\xi')\|_{L^2_{\xi'}} \\
&= \left\|(1 + |\xi'|^2)^{s/2} \int \mathcal{F}\widetilde{u}(\xi', \xi_d)\frac{1}{2\pi}\, d\xi_d\right\|_{L^2_{\xi'}}
\end{aligned}$$

Writing $|\xi|^2 = |\xi'|^2 + \xi_d^2$,

$$= \left\|\int \frac{(1 + |\xi'|^2)^{s/2}}{(1 + |\xi'|^2 + \xi_d^2)^{1/2}}((1 + |\xi'|^2 + \xi_d^2)^{1/2}\mathcal{F}\widetilde{u})\frac{1}{2\pi}\, d\xi_d\right\|_{L^2_{\xi'}}$$

Applying Cauchy-Schwarz,

$$\leq \left\| \left(\int \frac{(1+|\xi'|^2)^s}{1+|\xi'|^2+\xi_d^2}\, d\xi_d \right)^{1/2} \|(1+|\xi|'|^2+\xi_d^2)^{1/2}\mathcal{F}\widetilde{u}\|_{L^2_{\xi_d}} \right\|_{L^2_{\xi'}}$$

$$\leq \sup_{\xi'\in\mathbb{R}^{d-1}} \left(\int \frac{(1+|\xi'|^2)^s}{1+|\xi'|^2+\xi_d^2}\, d\xi_d \right)^{1/2} \underbrace{\|(1+|\xi|'|^2+\xi_d^2)^{1/2}\mathcal{F}\widetilde{u}\|_{L^2_{\xi_d}}\|_{L^2_\xi}}_{\|u\|_{H^1}}.$$

For what s is this supremum $< +\infty$? This is $s \leq 1/2$. $\square$

1.2 Extension from the boundary

It turns out that the image of $\mathrm{tr}_{\partial U}$ is *exactly* $H^{1/2}$.

Theorem 1.3 (Extension from ∂U)**.** *There exists a bounded linear map*

$$\mathrm{ext}_{\partial U} : H^{1/2}(\mathbb{R}^{d-1}) \to H^1(\mathbb{R}^d_+)$$

such that $\mathrm{tr}_{\partial U} \circ \mathrm{ext}_{\partial U} = \mathrm{id}$.

Proof. We will use the Poisson semigroup. Suppose we are given $g \in \mathcal{S}(\mathbb{R}^{d-1})$, and let $\eta \in C_c^\infty(\mathbb{R})$ be such that $\eta = 1$ for $|s| < 1$ and $\eta = 0$ for $|s| > 2$. Define $u = \mathrm{ext}_{\partial U}\, g$ by

$$\mathcal{F}_{x'}u(\xi', x^d) = \eta(x^d)e^{-x^d|\xi'|}\widehat{g}(\xi).$$

This right term is the solution to the Laplace equation on the half-space with boundary data g.

We need to show that

$$u \in H^1(\mathbb{R}^d_+) \iff \begin{array}{l} \text{(i) } u, \partial_1 u, \ldots, \partial_{d-1}u \in L^2 \\ \text{(ii) } \partial_d u \in L^2. \end{array}$$

(i) implies:

$$\begin{aligned} \|u\|_{L^2}^2 + \|\partial_1 u\|_{L^2}^2 + \cdots + \|\partial_{d-1}u\|_{L^2}^2 &= \|(1+|\xi'|^2)^{1/2}\mathcal{F}_{x'}u(\xi', x^d)\|^2_{L^2_{\xi'}L_{x^d}} \\ &= \|(1+|\xi'|^2)^{1/2}\eta(x^d)e^{-x^d|\xi'|}\widehat{g}(\xi')\|^2_{L^2_{\xi'}L^2_{x^d}} \end{aligned}$$

We can integrate in any order, so integrate the x^d integral first.

$$= \| \underbrace{(1+|\xi'|^2)^{1/4}\|\eta(x^d)e^{-x^d|\xi'|}\|_{L^2_{x^d}}}_{\text{NTS this is unif. bdd. } \xi' \in \mathbb{R}^{d-1}} (1+|\xi'|^2)^{1/4}\widehat{g}(\xi')\|^2_{L^2_{\xi'}}$$

We can use the bound

$$\|\eta(x^d)e^{-x^d|\xi'|}\|^2_{L^2_{x^d}} \lesssim 1,$$

and the substitution bound

$$\int \eta^2(x^d)e^{-2x^d|\xi'|}\,dx^d \lesssim \frac{1}{|\xi'|}.$$

This gives

$$\|\eta(x^d)e^{-x^d|\xi'|}\|_{L^2_{x^d}} \lesssim \min\left\{1, \tfrac{1}{|\xi'|^{1/2}}\right\} \lesssim (1+|\xi'|)^{-1/2}.$$

(ii) implies:

$$\begin{aligned}\partial_{x^d}u &= \partial_{x^d}(\eta(x^d)v), \qquad \mathcal{F}_{x'}v = e^{-x^d|\xi'|}\widehat{g}(\xi)\\ &= \eta'(x^d)v + \eta\partial_{x^d}v.\end{aligned}$$

The norm of the first term is bounded by $\|v\|_{L^2(x^d\in\operatorname{supp}\eta)}$, and the norm of the second term is

$$\begin{aligned}\|\eta\partial_{x^d}v\|_{L^2_{x'}L^2_{\xi_d}} &= \|\eta\partial_{x^d}(e^{-x^d|\xi'|}\widehat{g}(\xi'))\|_{L^2_{\xi'}L^2_{x^d}}\\ &= \||\xi'|\underbrace{e^{-x^d|\xi'|}\widehat{g}(\xi')\eta(x^d)}_{\mathcal{F}_{\xi'}u}\|_{L^2_{\xi'}L^2_{x^d}}\end{aligned}$$

Using (i),

$$\leq C\|g\|_{H^{1/2}}. \qquad \square$$

Remark 1.2. In fact, by the usual smooth partition of unity argument with boundary straightening, one can define $H^{1/2}(\partial U)$ for ∂U of class C^1 and prove the sharp trace theorem. The independence of this space from the smooth partition of unity and boundary straightening follows from interpolation theory (which you can find in the 1970 textbook of Stein).

Remark 1.3. For $p \neq 2$, $\operatorname{im}(\operatorname{tr}_{\partial U} W^{1,p}(U)) = B_p^{1-1/p,p}(\partial U)$. This is called the L^p**-Besov space of order** $1-1/p$ **and summability index** p. This is also covered in Stein's book.

1.3 The Gaglierdo-Nirenberg-Sobolev inequality and the Loomis-Whitney inequality

In a nutshell, Sobolev inequalities are a quantitative generalization of the fundamental theorem of calculus; we know the size of the derivative of a function, and we want to control the size of the function.

Theorem 1.4 (Gaglierdo-Nirenberg-Sobolev inequality)**.** *Let $d \geq 2$. For $u \in C_c^\infty(\mathbb{R}^d)$, we have*

$$\|u\|_{L^{\frac{d}{d-1}}}(\mathbb{R}^d) \leq C_d\|Du\|_{L^1(\mathbb{R}^d)},$$

where C_d is a constant depending only on d.

Remark 1.4. The exponent on the left hand side need not be remembered because it can be derived from scaling considerations (dimensional analysis). In particular, first observe that both sides are homogeneous: if $u \mapsto u_\lambda(x) = u(x/\lambda)$ for $\lambda > 0$, then

$$\begin{aligned}\|u_\lambda\|_{L^p} &= \left(\lambda^d \underbrace{\int \left|u\left(\frac{x}{\lambda}\right)\right|^p \frac{1}{\lambda^d}\,dx}_{=\int |u|^p\,dx'}\right)^{1/p} \\ &= \lambda^{d/p}\|u\|_{L^p}.\end{aligned}$$

On the other hand, $D(u_\lambda) = \frac{1}{\lambda}(Du)_\lambda$, so

$$\|D(u_\lambda)\|_{L^p} = \frac{1}{\lambda}\lambda^{d/p}\|Du\|_{L^p}.$$

Now compare these:

$$\begin{aligned}\|u_\lambda\|_{L^p} \le c\|Du_\lambda\|_{L^1} \qquad \forall\lambda > 0 &\iff \lambda^{d/p}\|u\|_{L^p} \le c\lambda^{-1+d}\|Du\|_{L^1} \qquad \forall\lambda > 0 \\ &\iff \frac{d}{p} = d-1 \\ &\iff p = \frac{d}{d-1}.\end{aligned}$$

All we are doing here is changing the unit of length and requiring that the inequality is invariant under our unit of length.

We will prove this next time. The key ingredient is another inequality. Denoting $(x^1, \dots, \widehat{x}^j, \cdots, x^d) = (x^1, \dots, x^{j-1}, x^{j+1}, \dots, x^d)$, we have the following.

Lemma 1.1 (Loomis-Whitney inequality)**.** *Let $d \ge 2$. For $j = 1, \dots, d$, suppose $f_j = f_j(x^1, \dots, \widehat{x}^j, \dots, x^d)$. Then*

$$\left\|\prod_{j=1}^d f_j\right\|_{L^1(\mathbb{R}^d)} \le \prod_{j=1}^d \|f_j\|_{L^{d-1}(\mathbb{R}^{d-1})}.$$

Proof. Integrate in each variable and apply Hölder:

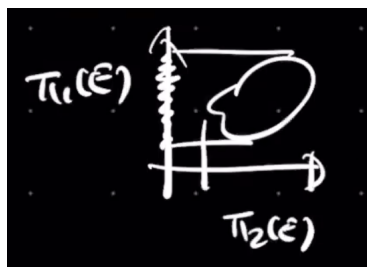
$$\begin{aligned}\int \left|\prod_{j=1}^d f_j\right| dx^1 &= |f_1| \int |f_2| \cdots |f_d|\,dx^1 \\ &\le |f_1| \|f_2\|_{L^{d-1}_{x^1}} \cdots \|f_d\|_{L^{d-1}_{x^1}}\end{aligned}$$

This is a function of $(x^2, \dots, x^d)$. Now integrate with respect to the next variable:

$$\iint \left| \prod_{j=1}^{d} f_j \right| dx^1\, dx^2 \leq \int |f_1| \|f_2\|_{L^{d-1}_{x^1}} \cdots \|f_d\|_{L^{d-1}_{x^1}}\, dx^2$$
$$= \|f_2\|_{L^{d-1}_{x^1}} \|f_1\|_{L^{d-1}_{x^2}} \|f_3\|_{L^{d-1}_{x^1,x^2}} \cdots \|f_d\|_{L^{d-1}_{x^1,x^2}}.$$

Iterating this gives the inequality. □

Remark 1.5. The Loomis-Whitney inequality answers the following geometric question. Suppose $E \subseteq \mathbb{R}^d$, and know the projections $\pi_j(E)$. Can we bound the measure of E by $|\pi_j(E)|$?



Yes!

$$\begin{aligned} |E| &= \int \mathbb{1}_E\, dx \\ &\leq \int \prod_{j=1}^{d} \mathbb{1}_{\pi_j(E)}(x^1, \dots, \widehat{x}^j, \dots, x^d)\, dx \\ &\overset{\text{L-W}}{\leq} \prod_{j=1}^{d} |\pi_j(E)|^{\frac{1}{d-1}}. \end{aligned}$$